Recall...

Virasoro algebra: $\mathrm{Vir} \overset{\text{as vect space}}{=} W \oplus \mathbb{C}Z$

Lie bracket $[L_m, L_n] = (m-n)L_{m+n} + \frac{Z}{12}(m^3 - m)\delta_{m+n,0}$

$[Z, L_n] = 0$

Unitary representation of Virasoro algebra:

Defn: A unitary rep. of Virasoro algebra is a Lie algebra homomorphism,

$\rho : \mathrm{Vir} \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$, where there exists a positive, semi-definite, Hermitian form on V:

$H : V \times V \longrightarrow \mathbb{C}$ s.t. we have the following identities:

$\boxed{L_n \cdot v \overset{\text{Denote}}{\equiv} \rho(L_n)(v)}$

$$H(\rho(L_n)v, w) = H(v, \rho(L_{-n})w)$$

$$H(\rho(Z)v, w) = H(v, \rho(Z)w)$$

Approach: 1) Forget about the H. Consider only the rep. of Virasoro.

2) Construct H

Defn: (Highest weight representation). A rep $\rho : Vir \to End(V)$ is called a <u>highest weight representation</u> if $\exists\, h, c \in \mathbb{C}$ and $v_0 \in V$ s.t.

$$Z \cdot v_0 = c v_0 ,$$

$$L_0 \cdot v_0 = h v_0 ,$$

$$L_n \cdot v_0 = 0 \quad \forall n \geq 1 ,$$

V generated by $\{ L_{-n_1}^{a_1}, \dots, L_{-n_k}^{a_k} \cdot v_0 \}$

Notation:

v_0 Highest weight vector

V Virasoro module with highest weight (c, h)

Remark: "Highest weight is actually the lowest weight"

"Largest possible highest weight module" = Verma module. $\mathcal{M}_{c,h}$

meaning $\varphi : \mathcal{M}_{c,h} \twoheadrightarrow V$ for any other highest weight module, V.

Defn | Construction:

$v_{n_1,\dots,n_k}$

$|||\to$ Denote

Basis: $\left\{ L_{-n_1} \cdots\cdots L_{-n_k} \cdot v_0 \right\}_{n_1 \geq n_2 \geq \cdots n_k > 0} \cup \{v_0\}$

Actions: $\rho(Z) = c \cdot \mathrm{id}_{M_{c,h}}$

$\rho(L_n) \cdot v_0 = 0 \quad \forall n > 0$

$\rho(L_0)\, v_0 = h$

$\rho(L_{-n}) = v_n$

$\rho(L_{\pm n})\, v_{n_1,\dots,n_k} \rightsquigarrow$ Use commutation relation & reorder if $n > n_1$

$$\rho(L_0)\, v_{n_1,\dots,n_k} = \left(\sum_{j=1}^{k} n_j + h \right) v_{n_1,\dots,n_k}$$

Exercise:

Check that the above is indeed a Lie alg. homomorphism

(i.e) $\quad [\rho(L_n), \rho(L_m)] = \rho\, [L_n, L_m]$

Remark:

Let V be a highest weight module. Then, $V = \bigoplus_{N \in \mathbb{N}} V_N$, where V_N is the eigenspace of $\rho(L_0)$ wrt eigenvalue $h+N$

Lemma:

Let V be a highest weight module. Let $U \subset V$ submodule. Then,

$$U = \bigoplus_{N \in \mathbb{N}} (V_N \cap U)$$

Proof:

$u \in U$ s.t.

$$\begin{array}{ccccc} u = & u_0 & + \cdots + & u_s \\ & \cap & & \cap \\ & V_0 & & V_s \end{array}$$

$$w = w_0 + \cdots + w_s$$

$$L_0 . w = h w_0 + \cdots + (s+h) w_s$$

$$(L_0)^2 w = h^2 w_0 + \cdots + (s+h)^2 w_s$$

$$\vdots$$

$$(L_0)^{s-1} w = h^{s-1} w_0 + \cdots + (s+h)^{s-1} w_s$$

Write it as matrix & show it's invertible

<u>Hermitian form on $\mathcal{M}_{c,h}$:</u>

$$\mathcal{M}_{c,h} = \bigoplus_{N \in \mathbb{N}} M_N$$ ($\because$ Verma module is a highest wt. rep too)

$\forall w \in \mathcal{M}_{c,h}$, it has a unique component $w_0 \in M_0$

$$w_0 = \underbrace{\langle w \rangle}_{\text{complex no.}} v_0$$

Define $H: M \times M \longrightarrow \mathbb{C}$ on basis $(v_{n_1,\dots,n_k}, v_0)$

$$H(v_{n_1,\dots,n_k}, v_{m_1,\dots,m_j}) = \langle L_{n_k} \cdots L_{n_1} L_{-m_1} \cdots L_{-m_j} v_0 \rangle$$

$$v_{n_1,\dots,n_k} = L_{-n_1} \cdots L_{-n_k} \cdot v_0$$

$H(L_n . v, w) = H(v, L_{-n} w)$ for unitary rep.

In particular $H(v_0, v_0) = 1$

$$H(v_0, v_{n_1,\dots,n_k}) = 0 = H(v_{n_1,\dots,n_k}, v_0)$$

Exercise:

If $c, h \in \mathbb{R}$, then $H(v, v') = H(v', v)$ for v, v' being basis vector.

$\forall w = \sum \lambda_j w_j \quad w' = \sum u_k w'_k$

basis vector

$$H(w, w') = \sum_{j,k} \bar{\lambda}_j u_j \; H(w_j, w_k)$$

Warning: H might be degenerate.

Thm: $h, c \in \mathbb{R}$

1) H defined above is the unique Hermitian form satisfying

$$H(v_0, v_0) = 1$$

$$H(v, L_{-n} w) = H(L_n v, w)$$

$$H(Zv, w) = H(v, Zw)$$

2) $H(v, w) = 0 \quad \forall v \in M_N, w \in M_m, m \neq N$

3) $\operatorname{Ker} H$ is $\underline{\text{THE}}$ maximal proper submodule of $\mathcal{M}_{c,h}$

Proof:

i) $H(v_{n_1,\dots,n_k}, v_{m_1,\dots,m_j}) = H'(v_0, L_{n_k}\cdots L_{n_1} v_{m_1,\dots,m_j})$

2) Direct computation.

3) $\operatorname{Ker} H := \{ v \in \mathcal{M}_{c,h} \mid H(w,v) = 0 \ \forall w \in \mathcal{M}_{c,h} \}$

(i) $\operatorname{Ker} H$ is a $\underline{\text{proper}}$ submodule.

$v_0 \notin \operatorname{Ker} H$

$\Rightarrow$ proper ✓

submodule $H(w, L_{-n} v) = H(L_n \cdot w, v) = 0$

(ii) Any submodule $U \subset \mathcal{M}_{c,h}$ $\qquad U \subset \operatorname{Ker} H$

if $\exists u \in U$,

$\exists v_{n_1,\dots,n_k}$ s.t. $H(v_{n_1,\dots,n_k}, u) \neq 0$

$\qquad\qquad = \langle L_{n_k}\cdots L_{n_1} u \rangle$

Goal: Determine whether H is positive (semi)-definite (depending on c, h)

Corollary:

If H is positive, semi-definite then $c, h \geq 0$

$$H(v_n, v_n) = 2nh + \frac{n^3 - n}{12} c$$

$n = 1 \rightsquigarrow h \geq 0$

General $\rightsquigarrow c \geq 0$

Kac Determinant:

V highest weight module $V = \bigoplus_N V_N$

$P(N) = \dim_{\mathbb{C}}(V_N)$, $\{b_1, \ldots, b_{p(N)}\}$

Matrix $(A^N_{i,j} = H(b_i, b_j))$

Example: $N = 2$. $\{v_2, v_{1,1}\}$

$$A^2 = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{pmatrix}$$

Thm: [Kac]

det A^N depends on (c, h) as follows:

$$\det A^N(c,h) = \underbrace{K_N}_{\substack{\text{constant} \\ \text{indep. of } c,h}} \prod_{\substack{p,q\in\mathbb{N} \\ pq\le N}} (h - h_{p,q}(c))^{P(N-pq)}$$

$$h_{p,q} = \frac{1}{48}\left((13-c)(p^2+q^2) + \sqrt{(c-1)(c-25)}(p^2-q^2) - 24pq - 2 + 2c\right)$$

<u>Thm:</u>

$\mathcal{M}_{c,h}$ is unitary (positive definite) for $c>1$, $h>0$

1) $\mathcal{M}_{c,h}$ —— " —— semi-definite for $c\ge 1$, $h\ge 0$

2) $\mathcal{M}_{c,h}$ unitary for $0\le c<1$, $h>0$ iff $\exists m\in\mathbb{N}$, $m>0$ so that $c=c(m)$ and

$$h_{p,q}(m) := \frac{((m+1)p - mq)^2}{4m(m+1)}, \quad m\in\mathbb{N}$$

$$c(m) := 1 - \frac{6}{m(m+1)}, \quad m\in\mathbb{N}\setminus\{1\}$$

$\mathcal{M}_{c,h}$ unitary (positive, semi-definite)
Then

$$W_{c,h} = \mathcal{M}_{c,h} / \mathrm{Ker} H \quad \text{positive def.}$$

<u>Thm</u>: 1) $\mathcal{M}_{c,h}$ is indecomposable (works for all highest weight module)

2) If $\mathcal{M}_{c,h}$ reducible, then $\exists$ maximal proper submodule $I_{c,h}$ s.t. $\mathcal{M}_{c,h}/I_{c,h}$ being irreducible highest weight module.

3) Any positive definite, unitary heighest weight module is irreducible

<u>Proj. rep. of $\mathrm{Diff}_+(S^1)$:</u>

<u>Recall</u>... Take $\mathrm{Lie}(\mathrm{Diff}_+(S^1)) = \underset{\text{vector field}}{\mathrm{Vect}}(S^1)$

Complexify it $\quad \mathrm{Vect}^{\mathbb{C}}(S^1) = \mathrm{Vect}(S^1) \otimes_{\mathbb{R}} \mathbb{C}$

$W \subset \mathrm{Vect}^{\mathbb{C}}(S^1)$ of polynomial vector fields.

Generators $\quad L_n = z^{1-n} \dfrac{d}{dz}$

$\rightsquigarrow$ Virasoro algebra is the unique non-trivial extension of W by $\mathbb{C}$

$$U : \mathrm{Diff}_+(S^1) \longrightarrow U(\mathbb{P}(\underset{\substack{\uparrow \\ \text{some Hilbert space (we need to find } \mathbb{H})}}{\mathbb{H}}))$$

$\mathbb{H} :=$ completion of $W_{c,h}$

$$W_{c,h} \subset \overline{W_{c,h}} \subset \mathbb{H}$$

$$\rho_{c,h} : \mathrm{Vir} \longrightarrow \mathrm{End}_{\mathbb{C}}(W_{c,h})$$

$$W_{c,h} \subset \overline{W_{c,h}} \subset \mathbb{H}$$

$$\curvearrowleft \overline{\rho}_{c,h}$$

$\exists\, U_{c,h} : \mathrm{Diff}_+(S^1) \longrightarrow U(\mathbb{P}(\mathbb{H}))$ s.t.

$\forall \xi \in \mathrm{Vect}(S^1)$

$$\hat{\gamma}\left(\exp\left(\overline{\rho}_{c,h}(\xi)\right)\right) = U_{c,h}(\exp(\xi))$$